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The sixth and eighth moments of Fourier coefficients of cusp forms ☆

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A R T I C L E   I N F O

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A B S T R A C T

Let $\lambda(n)$ be the nth normalized Fourier coefficient of a holomorphic Hecke eigencuspform $f(z)$ of even integral weight k for the full modular group. In this paper we are able to prove the following results.

(i) For any $\varepsilon > 0$, we have

$$\sum_{n\leqslant x}\lambda^6(n) = xP_1(\log x) + O_{f,\varepsilon}\big(x^{\frac{31}{32}+\varepsilon}\big),$$

where $P_1(x)$ is a polynomial of degree 4.

(ii) For any $\varepsilon > 0$, we have

$$\sum_{n\leqslant x}\lambda^8(n) = xP_2(\log x) + O_{f,\varepsilon}\big(x^{\frac{127}{128}+\varepsilon}\big),$$

where $P_2(x)$ is a polynomial of degree 13.

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1. Introduction and main results

Let $S_k(\Gamma)$ be the space of holomorphic cusp forms of even integral weight k for the full modular group $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Suppose that $f(z)$ is an eigenfunction of all Hecke operators belonging to $S_k(\Gamma)$. Then the Hecke eigencuspform $f(z)$ has the following Fourier expansion at the cusp ∞

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$$f(z)=\sum_{n=1}^{\infty}a(n)e^{2\pi inz},$$

where we normalize $f(z)$ such that $a(1)=1$. Instead of $a(n)$, one often considers the normalized Fourier coefficient

$$\lambda(n)=\frac{a(n)}{n^{\frac{k-1}{2}}}.$$

Then from the theory of Hecke operators, $\lambda(n)$ is real and satisfies the multiplicative property

$$\lambda(m)\lambda(n)=\sum_{d|(m,n)}\lambda\left(\frac{mn}{d^2}\right), \tag{1.1}$$

where $m\geqslant 1$ and $n\geqslant 1$ are any integers.

The Fourier coefficients of cusp forms are interesting objects. In 1974, P. Deligne [2] proved the Ramanujan–Petersson conjecture

$$\left|\lambda(n)\right|\leqslant d(n), \tag{1.2}$$

where $d(n)$ is the divisor function. As a corollary, he proved that for any $\varepsilon>0$,

$$S(x)=\sum_{n\leqslant x}\lambda(n)\ll_f x^{\frac{1}{3}+\varepsilon}.$$

In 1989, Hafner and Ivic' [6] were able to remove the factor x^{ε} of Deligne's result, i.e.

$$S(x)=\sum_{n\leqslant x}\lambda(n)\ll_f x^{\frac{1}{3}}.$$

In this direction, the best known result is due to Rankin [15]

$$S(x)=\sum_{n\leqslant x}\lambda(n)\ll_f x^{\frac{1}{3}}(\log x)^{-\delta},$$

where $0<\delta<0.06$.

Rankin [14] and Selberg [16] invented the powerful Rankin–Selberg method, and then successfully showed that

$$\sum_{n\leqslant x}\lambda^2(n)=cx+O_{f,\varepsilon}\bigl(x^{\frac{3}{5}+\varepsilon}\bigr).$$

Later based on the works about symmetric power L-functions, Moreno and Shahidi [13] were able to prove

$$\sum_{n\leqslant x}\tau_0^4(n)\sim cx\log x,\quad x\to\infty,$$

where $\tau_0(n)=\tau(n)/n^{\frac{11}{2}}$ is the normalized Ramanujan tau-function. Obviously Moreno and Shahidi's result also holds true if we replace $\tau_0(n)$ by the normalized Fourier coefficient $\lambda(n)$. In 2001, Fomenko

[3] improved Moreno and Shahidi's result by showing that

$$\sum_{n\leqslant x}\lambda^4(n)=cx\log x+c'x+O_{f,\varepsilon}\big(x^{\frac{9}{10}+\varepsilon}\big).$$

Very recently, inspired by the beautiful paper of Friedlander and Iwaniec [4], Lü [12] proved that

$$\sum_{n\leqslant x}\lambda^4(n)=cx\log x+c'x+O_{f,\varepsilon}\big(x^{\frac{7}{8}+\varepsilon}\big).$$

In this paper we are interested in the sixth and eighth moments of Fourier coefficients of cusp forms. Thanking to the important results about symmetric power L-functions and their Rankin–Selberg L-functions (see, for example, [5,8–11,17]), we are able to show the following results.

Theorem 1.1. *Let $f(z)\in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k for the full modular group, and $\lambda(n)$ denote its nth normalized Fourier coefficient. Then for any $\varepsilon>0$, we have*

$$\sum_{n\leqslant x}\lambda^6(n)=xP_1(\log x)+O_{f,\varepsilon}\big(x^{\frac{31}{32}+\varepsilon}\big),$$

where $P(x)$ is a polynomial of degree 4.

Theorem 1.2. *Let $f(z)\in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k for the full modular group, and $\lambda(n)$ denote its nth normalized Fourier coefficient. Then for any $\varepsilon>0$, we have*

$$\sum_{n\leqslant x}\lambda^8(n)=xP_2(\log x)+O_{f,\varepsilon}\big(x^{\frac{127}{128}+\varepsilon}\big),$$

where $P_2(x)$ is a polynomial of degree 13.

2. Some lemmas

In this section we recall or establish some results, which we shall use in the proof of our main results.

Lemma 2.1. *Let $f(z)\in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k for the full modular group, and $\lambda(n)$ denote its nth normalized Fourier coefficient. We introduce*

$$L(s)=\sum_{n=1}^{\infty}\frac{\lambda^6(n)}{n^s}, \tag{2.1}$$

for $\operatorname{Re}s>1$. For $j=2,3,4$, let $L(\operatorname{sym}^j f,s)$ be the jth symmetric power L-function associated to f, and $L(\operatorname{sym}^j f\times\operatorname{sym}^j f,s)$ be the Rankin–Selberg L-function associated to $\operatorname{sym}^j f$ and $\operatorname{sym}^j f$.

Then we have that for $\operatorname{Re}s>1$,

$$L(s)=\zeta^4(s)L^8\big(\operatorname{sym}^2 f,s\big)L^4\big(\operatorname{sym}^4 f,s\big)L\big(\operatorname{sym}^3 f\times\operatorname{sym}^3 f,s\big)U(s), \tag{2.2}$$

where $\zeta(s)$ is the Riemann zeta-function, and $U(s)$ is a Dirichlet series, which converges uniformly and absolutely in the half plane $\operatorname{Re}s\geqslant 1/2+\varepsilon$ for any $\varepsilon>0$.

Proof. According to Deligne [2], for any prime number p there are $\alpha(p)$ and $\beta(p)$ such that

$$\lambda(p)=\alpha(p)+\beta(p) \quad \text{and} \quad \left|\alpha(p)\right|=\alpha(p)\beta(p)=1. \tag{2.3}$$

For $\operatorname{Re} s>1$, the Riemann zeta-function is defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_p\left(1+\frac{1}{p^s}+\cdots+\frac{1}{p^{ks}}+\cdots\right). \tag{2.4}$$

The jth symmetric power L-function attached to $f\in S_k(\Gamma)$ is defined by

$$L\left(\operatorname{sym}^j f,s\right):=\prod_p\prod_{m=0}^{j}\left(1-\alpha(p)^{j-m}\beta(p)^m p^{-s}\right)^{-1} \tag{2.5}$$

for $\operatorname{Re} s>1$. The product over primes gives a Dirichlet series representation for $L(\operatorname{sym}^j f,s)$: for $\operatorname{Re} s>1$,

$$L\left(\operatorname{sym}^j f,s\right)=\sum_{n=1}^{\infty}\frac{\lambda_{\operatorname{sym}^j f}(n)}{n^s},$$

where $\lambda_{\operatorname{sym}^j f}(n)$ is a multiplicative function. From (2.3), we have

$$\left|\lambda_{\operatorname{sym}^j f}(n)\right|\leqslant d_{j+1}(n), \tag{2.6}$$

where $d_k(n)$ is the nth coefficient of the Dirichlet series $\zeta^k(s)$.

The Rankin–Selberg L-function associated to $\operatorname{sym}^j f$ and $\operatorname{sym}^j f$ is defined by

$$L\left(\operatorname{sym}^j f\times\operatorname{sym}^j f,s\right):=\prod_p\prod_{m=0}^{j}\prod_{u=0}^{j}\left(1-\alpha(p)^{j-m}\beta(p)^m\alpha(p)^{j-u}\beta(p)^u p^{-s}\right)^{-1} \tag{2.7}$$

for $\operatorname{Re} s>1$. The product over primes also gives a Dirichlet series representation for $L(\operatorname{sym}^j f\times\operatorname{sym}^j f,s)$: for $\operatorname{Re} s>1$,

$$L\left(\operatorname{sym}^j f\times\operatorname{sym}^j f,s\right)=\sum_{n=1}^{\infty}\frac{\lambda_{\operatorname{sym}^j f\times\operatorname{sym}^j f}(n)}{n^s},$$

where $\lambda_{\operatorname{sym}^j f\times\operatorname{sym}^j f}(n)$ is a multiplicative function. From (2.3), we have

$$\left|\lambda_{\operatorname{sym}^j f\times\operatorname{sym}^j f}(n)\right|\leqslant d_{(j+1)^2}(n). \tag{2.8}$$

Then we have that for $\operatorname{Re} s>1$,

$$L\left(\operatorname{sym}^j f,s\right)=\prod_p\left(1+\frac{\lambda_{\operatorname{sym}^j f}(p)}{p^s}+\cdots+\frac{\lambda_{\operatorname{sym}^j f}(p^k)}{p^{ks}}+\cdots\right), \tag{2.9}$$

and

$$L\big(\mathrm{sym}^j f \times \mathrm{sym}^j f, s\big) = \prod_p \left(1 + \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p)}{p^s} + \cdots + \frac{\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p^k)}{p^{ks}} + \cdots\right). \tag{2.10}$$

From (2.5), (2.7), (2.9), and (2.10), we have

$$\lambda_{\mathrm{sym}^j}(p) = \sum_{m=0}^{j} \alpha(p)^{j-m}\beta(p)^m, \tag{2.11}$$

and

$$\lambda_{\mathrm{sym}^j f \times \mathrm{sym}^j f}(p) = \sum_{m=0}^{j}\sum_{u=0}^{j} \alpha(p)^{j-m}\beta(p)^m \alpha(p)^{j-u}\beta(p)^u = \left(\sum_{m=0}^{j} \alpha(p)^{j-m}\beta(p)^m\right)^2. \tag{2.12}$$

For $\operatorname{Re} s > 1$, we can write $\zeta^4(s)L^8(\mathrm{sym}^2 f, s)L^4(\mathrm{sym}^4 f, s)L(\mathrm{sym}^3 f \times \mathrm{sym}^3 f, s)$ as an Euler product

$$\zeta^4(s)L^8\big(\mathrm{sym}^2 f, s\big)L^4\big(\mathrm{sym}^4 f, s\big)L\big(\mathrm{sym}^3 f \times \mathrm{sym}^3 f, s\big) =: \prod_p \left(1 + \frac{b(p)}{p^s} + \cdots + \frac{b(p^k)}{p^{ks}} + \cdots\right). \tag{2.13}$$

From (2.4), (2.9), (2.10), we have

$$b(p) = 4 + 8\lambda_{\mathrm{sym}^2}(p) + 4\lambda_{\mathrm{sym}^4}(p) + \lambda_{\mathrm{sym}^3 f \times \mathrm{sym}^3 f}(p). \tag{2.14}$$

From (2.3), (2.11), and (2.12), it is easy to check that

$$\begin{aligned} b(p) &= 4 + 8\big(\alpha(p)^2 + 1 + \beta(p)^2\big) + 4\big(\alpha(p)^4 + \alpha(p)^2 + 1 + \beta(p)^2 + \beta(p)^4\big) \\ &\quad + \big(\alpha(p)^3 + \alpha(p) + \beta(p) + \beta(p)^3\big)^2 \\ &= \big(\alpha(p) + \beta(p)\big)^6 = \lambda^6(p). \end{aligned} \tag{2.15}$$

On the other hand, from (1.2) we learn that

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda^6(n)}{n^s} \tag{2.16}$$

is absolutely convergent in the half plane $\operatorname{Re} s > 1$. On noting that $\lambda^6(n)$ is a multiplicative function, we have that for $\operatorname{Re} s > 1$,

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda^6(n)}{n^s} = \prod_p \left(1 + \frac{\lambda^6(p)}{p^s} + \frac{\lambda^6(p^2)}{p^{2s}} + \cdots + \frac{\lambda^6(p^k)}{p^{ks}} + \cdots\right). \tag{2.17}$$

Therefore from (2.13), (2.15), and (2.17), we have that for $\operatorname{Re} s > 1$,

$$\begin{aligned} L(s) &= \zeta^4(s)L^8\big(\mathrm{sym}^2 f, s\big)L^4\big(\mathrm{sym}^4 f, s\big)L\big(\mathrm{sym}^3 f \times \mathrm{sym}^3 f, s\big) \prod_p \left(1 + \frac{\lambda^6(p^2) - b(p^2)}{p^{2s}} + \cdots\right) \\ &=: \zeta^4(s)L^8\big(\mathrm{sym}^2 f, s\big)L^4\big(\mathrm{sym}^4 f, s\big)L\big(\mathrm{sym}^3 f \times \mathrm{sym}^3 f, s\big)U(s). \end{aligned}$$

From (1.2), (2.6), and (2.8), it is obvious that $U(s)$ converges uniformly and absolutely in the half plane $\operatorname{Re} s \geqslant \frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k for the full modular group, and $\lambda(n)$ denote its nth normalized Fourier coefficient. We introduce*

$$L'(s) = \sum_{n=1}^{\infty} \frac{\lambda^8(n)}{n^s},$$

for $\operatorname{Re} s > 1$.

Then we have that for $\operatorname{Re} s > 1$,

$$L'(s) = \zeta^7(s) L^{21}(\operatorname{sym}^2 f, s) L^{13}(\operatorname{sym}^4 f, s) L^6(\operatorname{sym}^3 f \times \operatorname{sym}^3 f, s) L(\operatorname{sym}^4 f \times \operatorname{sym}^4 f, s) U'(s),$$

where $U'(s)$ is a Dirichlet series, which converges uniformly and absolutely in the half plane $\operatorname{Re} s \geqslant 1/2 + \varepsilon$ *for any* $\varepsilon > 0$.

Proof. On noting that

$$\begin{aligned}(\alpha(p) + \beta(p))^8 &= 7 + 21(\alpha(p)^2 + 1 + \beta(p)^2) + 13(\alpha(p)^4 + \alpha(p)^2 + 1 + \beta(p)^2 + \beta(p)^4) \\ &\quad + 6(\alpha(p)^3 + \alpha(p) + \beta(p) + \beta(p)^3)^2 + (\alpha(p)^4 + \alpha(p)^2 + 1 + \beta(p)^2 + \beta(p)^4)^2,\end{aligned}$$

we can follow the same way as that used in Lemma 2.1 to establish this lemma. □

Before we go further, we give a brief account of the key point in the proof of Lemmas 2.1 and 2.2. Let $t = \alpha(p) + \beta(p)$. The polynomials S_j for the trace of symmetric jth power are defined by

$$\begin{aligned}
&S_0 = 1;\\
&S_1 = \alpha(p) + \beta(p) = t;\\
&S_2 = \alpha(p)^2 + 1 + \beta(p)^2 = t^2 - 1;\\
&S_3 = \alpha(p)^3 + \alpha(p) + \beta(p) + \beta(p)^3 = t^3 - 2t;\\
&S_4 = \alpha(p)^4 + \alpha(p)^2 + 1 + \beta(p)^2 + \beta(p)^4 = t^4 - 3t^2 + 1;\\
&S_5 = \alpha(p)^5 + \alpha(p)^3 + \alpha(p) + 1 + \beta(p) + \beta(p)^3 + \beta(p)^5 = t^5 - 4t^3 + 3t;\\
&S_6 = \alpha(p)^6 + \alpha(p)^4 + \alpha(p)^2 + 1 + \beta(p)^2 + \beta(p)^4 + \beta(p)^6 = t^6 - 5t^4 + 6t^2 - 1;\\
&S_8 = \alpha(p)^8 + \alpha(p)^6 + \alpha(p)^4 + \alpha(p)^2 + 1 + \beta(p)^2 + \beta(p)^4 + \beta(p)^6 + \beta(p)^8\\
&\quad = t^8 - 7t^6 + 15t^4 - 10t^2 + 1.
\end{aligned}$$

Then

$$t^6 = 5 + 9S_2 + 5S_4 + S_6; \qquad t^8 = 14 + 28S_2 + 20S_4 + 7S_6 + S_8.$$

On the other hand, we have

$$S_3^2 = 1 + S_2 + S_4 + S_6; \qquad S_4^2 = 1 + S_2 + S_4 + S_6 + S_8.$$

Therefore

$$t^6 = 4 + 8S_2 + 4S_4 + S_3^2; \qquad t^8 = 7 + 21S_2 + 13S_4 + 6S_3^2 + S_4^2.$$

In addition, by Hecke relations we have

$$S_j = \frac{\alpha(p)^{j+1} - \beta(p)^{j+1}}{\alpha(p) - \beta(p)} = \sum_{m=0}^{j} \alpha(p)^{j-m}\beta(p)^m = \lambda(p^j), \quad j \geqslant 0.$$

Then we have

$$\lambda^6(p) = 4 + 8\lambda(p^2) + 4\lambda(p^4) + \lambda^2(p^3);$$
$$\lambda^8(p) = 7 + 21\lambda(p^2) + 13\lambda(p^4) + 6\lambda^2(p^3) + \lambda^2(p^4).$$

In essence, these two identities determine Lemmas 2.1 and 2.2.

For the cases of holomorphic modular forms considered in this paper, Cogdell and Michel [1], Lau and Wu [11] gave the explicit version of the famous results established in Gelbart and Jacquet [5], Kim and Shahidi [9,10], and Kim [8], which state that for $j = 2, 3, 4$, $L(\mathrm{sym}^j f, s)$ and $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ have meromorphic continuations to the whole complex plane $\mathbb{C}$, and satisfy certain functional equations.

Lemma 2.3. *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k. The L-function associated to $\mathrm{sym}^j f$ is defined in (2.5). For $j = 2, 3, 4$, the archimedean local factor of $L(\mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f, s) = \begin{cases} \prod_{\nu=0}^{n} \Gamma_{\mathbb{C}}(s + (\nu + \frac{1}{2})(k-1)), & \text{if } j = 2n+1, \\ \Gamma_{\mathbb{R}}(s + \delta_{2\nmid n}) \prod_{\nu=1}^{n} \Gamma_{\mathbb{C}}(s + \nu(k-1)), & \text{if } j = 2n, \end{cases}$$

where $\Gamma_{\mathbb{R}} = \pi^{-s/2}\Gamma(s/2)$, $\Gamma_{\mathbb{C}} = 2(2\pi)^{-s}\Gamma(s)$, and

$$\delta_{2\nmid n} = \begin{cases} 1, & \text{if } 2 \nmid n, \\ 0, & \text{otherwise}. \end{cases}$$

For $2 \leqslant j \leqslant 4$, it is known that the complete L-function

$$\Lambda(\mathrm{sym}^j f, s) = L_\infty(\mathrm{sym}^j f, s) L(\mathrm{sym}^j f, s)$$

is an entire function on the whole complex plane $\mathbb{C}$, and satisfies the functional equation

$$\Lambda(\mathrm{sym}^j f, s) = \epsilon_{\mathrm{sym}^j f} \Lambda(\mathrm{sym}^j f, 1-s),$$

where $\epsilon_{\mathrm{sym}^j f} = \pm 1$.

Proof. See Section 3.2.1 of Cogdell and Michel [1]. □

Lemma 2.4. *Let $f(z) \in S_k(\Gamma)$ be a Hecke eigencuspform of even integral weight k. The Rankin–Selberg L-function associated to $\mathrm{sym}^j f$ and $\mathrm{sym}^j f$ is defined in (2.7). For $j = 2, 3, 4$, the archimedean local factor of $L(\mathrm{sym}^j f \times \mathrm{sym}^j f, s)$ is*

$$L_\infty(\mathrm{sym}^j f \times \mathrm{sym}^j f, s) = \Gamma_{\mathbb{R}}(s)^{\delta_{2|j}} \Gamma_{\mathbb{C}}(s)^{[j/2]+\delta_{2\nmid j}} \prod_{\nu=1}^{j} \Gamma_{\mathbb{C}}(s + \nu(k-1))^{j-\nu+1},$$

where

$$\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2),\quad \Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma(s),\quad \delta_{2|j}=1-\delta_{2\nmid j},\quad \textit{and}\quad \delta_{2\nmid j}=\begin{cases}1, & \textit{if } 2\nmid j,\\ 0, & \textit{otherwise}.\end{cases}$$

Then the complete L-function

$$\Lambda\big(\mathrm{sym}^j f\times\mathrm{sym}^j f,s\big)=:L_\infty\big(\mathrm{sym}^j f\times\mathrm{sym}^j f,s\big)L\big(\mathrm{sym}^j f\times\mathrm{sym}^j f,s\big)$$

is entire except for simple poles at $s=0,1$ and satisfies the functional equation

$$\Lambda\big(\mathrm{sym}^j f\times\mathrm{sym}^j f,s\big)=\epsilon_{\mathrm{sym}^j f\times\mathrm{sym}^j f}\Lambda\big(\mathrm{sym}^j f\times\mathrm{sym}^j f,1-s\big)$$

with $|\epsilon_{\mathrm{sym}^j f\times\mathrm{sym}^j f}|=1$.

Proof. This is Proposition 2.1 in Lau and Wu [11]. $\square$

Lemma 2.5. *Let $j=2,3,4$. Then for any $\varepsilon>0$ and $0\leqslant\sigma\leqslant1$, we have*

$$L\big(\mathrm{sym}^j f,\sigma+it\big)\ll_{f,\varepsilon}\big(1+|t|\big)^{\frac{j+1}{2}(1-\sigma)+\varepsilon},$$

and

$$L\big(\mathrm{sym}^j f\times\mathrm{sym}^j f,\sigma+it\big)\ll_{f,\varepsilon}\big(1+|t|\big)^{\frac{(j+1)^2}{2}(1-\sigma)+\varepsilon}.$$

Proof. From Lemmas 2.3 and 2.4, we can follow standard arguments to establish the convexity bounds for $L(\mathrm{sym}^j f,\sigma+it)$ and $L(\mathrm{sym}^j f\times\mathrm{sym}^j f,\sigma+it)$ in the critical strip $\frac{1}{2}\leqslant\sigma\leqslant1$ (see for example, Chapter 5 of [7]). $\square$

Lemma 2.6. *Let $L(f,s)$ be a Dirichlet series with Euler product of degree $m\geqslant2$, which means*

$$L(f,s)=\sum_{n=1}^{\infty}\lambda_f(n)n^{-s}=\prod_{p<\infty}\prod_{j=1}^{m}\left(1-\frac{\alpha_f(p,j)}{p^s}\right)^{-1},$$

where $\alpha_f(p,j)$, $j=1,\dots,m$, are the local parameters of $L(f,s)$ at prime p and $\lambda_f(n)\ll n^{\varepsilon}$. Assume that this series and its Euler product are absolutely convergent for $\operatorname{Re}s>1$. Assume also that it is entire except possibly for simple poles at $s=0,1$, and satisfies a functional equation of Riemann type. Then we have that for $T\geqslant1$,

$$\int_T^{2T}\big|L(f,1/2+\varepsilon+it)\big|^2\,dt\ll T^{\frac{m}{2}+\varepsilon}.$$

Proof. This is a well-known folklore result. $\square$

3. Proof of the theorems

In this section we give the proof of Theorem 1.1. The proof of Theorem 1.2 is similar to that of Theorem 1.1. In order to avoid repetition, we omit the proof of Theorem 1.2.

Recall that we have defined

$$L(s)=\sum_{n=1}^{\infty}\frac{\lambda^6(n)}{n^s}, \tag{3.1}$$

for $\operatorname{Re} s>1$. From Lemmas 2.1, 2.3, and 2.4, we learn that

$$L(s)=\zeta^4(s)L^8\big(\mathrm{sym}^2 f,s\big)L^4\big(\mathrm{sym}^4 f,s\big)L\big(\mathrm{sym}^3 f\times\mathrm{sym}^3 f,s\big)U(s)$$

can be analytically continued to the half plane $\operatorname{Re} s>1/2$. In this region, $L(s)$ only has a pole $s=1$ of order 5.

By (3.1) and Perron's formula (see Proposition 5.54 in [7]), we have

$$\sum_{n\leqslant x}\lambda(n)^6=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}L(s)\frac{x^s}{s}\,ds+O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.2}$$

where $b=1+\varepsilon$ and $1\leqslant T\leqslant x$ is a parameter to be chosen later. Here we have used (1.2).

Then we move the integration to the parallel segment with $\operatorname{Re} s=\frac{1}{2}+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}\sum_{n\leqslant x}\lambda(n)^6 &= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT}+\int_{\frac{1}{2}+\varepsilon+iT}^{b+iT}+\int_{b-iT}^{\frac{1}{2}+\varepsilon-iT}\right\}L(s)\frac{x^s}{s}\,ds+\operatorname{Res}_{s=1}\left\{L(s)\frac{x^s}{s}\,ds\right\}+O\left(\frac{x^{1+\varepsilon}}{T}\right)\\ &=: xP_1(\log x)+J_1+J_2+J_3+O\left(\frac{x^{1+\varepsilon}}{T}\right),\end{aligned} \tag{3.3}$$

where $P_1(x)$ is a polynomial of degree 4.

To go further, we recall that $\zeta^4(s)L^8(\mathrm{sym}^2 f,s)L^4(\mathrm{sym}^4 f,s)L(\mathrm{sym}^3 f\times\mathrm{sym}^3 f,s)$ is a Riemann-type nice L-function with Euler product of degree $m=64$.

For J_1, from Lemma 2.1 we have

$$J_1\ll x^{\frac{1}{2}+\varepsilon}\int_1^T\big|\big\{\zeta^4(s)L^8\big(\mathrm{sym}^2 f,s\big)L^4\big(\mathrm{sym}^4 f,s\big)L\big(\mathrm{sym}^3 f\times\mathrm{sym}^3 f,s\big)\big\}\big|_{s=1/2+\varepsilon+it}\big|t^{-1}\,dt+x^{\frac{1}{2}+\varepsilon}.$$

Then by Cauchy's inequality, we have

$$\begin{aligned}J_1 &\ll x^{1/2+\varepsilon}\log T\max_{T_1\leqslant T}\left\{\frac{1}{T_1}\left(\int_{T_1/2}^{T_1}\big|\big\{\zeta^4(s)L^8\big(\mathrm{sym}^2 f,s\big)L^4\big(\mathrm{sym}^4 f,s\big)\big\}\big|_{s=1/2+\varepsilon+it}\big|^2\,dt\right)^{\frac{1}{2}}\right.\\ &\quad\left.\times\left(\int_{T_1/2}^{T_1}\big|L\big(\mathrm{sym}^3 f\times\mathrm{sym}^3 f,1/2+\varepsilon+it\big)\big|^2\,dt\right)^{\frac{1}{2}}\right\}+x^{\frac{1}{2}+\varepsilon}\\ &\ll x^{\frac{1}{2}+\varepsilon}T^{15+\varepsilon},\end{aligned} \tag{3.4}$$

where we have used Lemma 2.6 in the following forms

$$\int_{T_1/2}^{T_1} \left|\left\{\zeta^4(s)L^8\left(\mathrm{sym}^2 f, s\right)L^4\left(\mathrm{sym}^4 f, s\right)\right\}\right|_{s=1/2+\varepsilon+it}\right|^2 dt \ll T^{24+\varepsilon},$$

and

$$\int_{T_1/2}^{T_1} \left|L\left(\mathrm{sym}^3 f \times \mathrm{sym}^3 f, 1/2+\varepsilon+it\right)\right|^2 dt \ll T^{8+\varepsilon}.$$

For the integral over the horizontal segments, we use Lemma 2.5, and the trivial bound for Riemann zeta-function

$$\left|\zeta(\sigma+it)\right| \ll \left(|t|+1\right)^{\frac{1-\sigma}{2}+\varepsilon}$$

to bound

$$\begin{aligned}
J_2+J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^{b} x^{\sigma}\left|\left\{\zeta^4(s)L^8\left(\mathrm{sym}^2 f, s\right)L^4\left(\mathrm{sym}^4 f, s\right)L\left(\mathrm{sym}^3 f \times \mathrm{sym}^3 f, s\right)\right\}\right|_{s=\sigma+iT}\right|T^{-1}\, d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant b} x^{\sigma}T^{32(1-\sigma)+\varepsilon}T^{-1} = \max_{\frac{1}{2}+\varepsilon\leqslant\sigma\leqslant b}\left(\frac{x}{T^{32}}\right)^{\sigma}T^{31+\varepsilon} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon}T^{15+\varepsilon}.
\end{aligned} \tag{3.5}$$

From (3.3), (3.4) and (3.5), we have

$$\sum_{n\leqslant x}\lambda^6(n) = xP_1(\log x) + O\left(\frac{x^{1+\varepsilon}}{T}\right) + O\left(x^{\frac{1}{2}+\varepsilon}T^{15+\varepsilon}\right). \tag{3.6}$$

On taking $T = x^{\frac{1}{32}}$ in (3.6), we have

$$\sum_{n\leqslant x}\lambda^6(n) = xP_1(\log x) + O\left(x^{\frac{31}{32}+\varepsilon}\right). \tag{3.7}$$

This completes the proof of Theorem 1.1.

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